

## A NOTE ON EQUIMULTIPLE DEFORMATIONS

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ABSTRACT. While the tangent space to an equisingular family of curves can be described by the sections of a twisted ideal sheaf, this is no longer true if we only prescribe the multiplicity which a singular point should have. However, it is still possible to compute the dimension of the tangent space with the aid of the equimultiplicity ideal. In this note we consider families  $\mathcal{L}_m = \{(C, p) \in |L| \times S \mid \text{mult}_p(C) = m\}$  for some linear system  $|L|$  on a smooth projective surface  $S$  and a fixed positive integer  $m$ , and we compute the dimension of the tangent space to  $\mathcal{L}_m$  at a point  $(C, p)$  depending on whether  $p$  is a unitangential singular point of  $C$  or not. We deduce that the expected dimension of  $\mathcal{L}_m$  at  $(C, p)$  in any case is just  $\dim |L| - \frac{m \cdot (m+1)}{2} + 2$ . The result is used in the study of triple-point defective surfaces in [ChM06a] and [ChM06b].

The paper is based on considerations about the Hilbert scheme of curves in a projective surface (see e.g. [Mum66], Lecture 22) and about local equimultiple deformations of plane curves (see [Wah74]).

**Definition 1**

Let  $T$  be a complex space. An *embedded family of curves in  $S$  with section* over  $T$  is a commutative diagram of morphisms

$$\begin{array}{ccc} & \mathcal{C} & \hookrightarrow T \times S \\ \sigma \curvearrowright & \downarrow \varphi & \swarrow \\ & T & \end{array}$$

where  $\text{codim}_{T \times S}(\mathcal{C}) = 1$ ,  $\varphi$  is flat and proper, and  $\sigma$  is a section, i.e.  $\varphi \circ \sigma = \text{id}_T$ . Thus we have a morphism  $\mathcal{O}_T \rightarrow \varphi_* \mathcal{O}_{\mathcal{C}} = \varphi_*(\mathcal{O}_{T \times S} / \mathcal{I}_{\mathcal{C}})$  such that  $\varphi_* \mathcal{O}_{\mathcal{C}}$  is a flat  $\mathcal{O}_T$ -module.

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The family is said to be *equimultiple* of multiplicity  $m$  *along the section*  $\sigma$  if the ideal sheaf  $\mathcal{I}_{\mathcal{C}}$  of  $\mathcal{C}$  in  $\mathcal{O}_{T \times S}$  satisfies

$$\mathcal{I}_{\mathcal{C}} \subseteq \mathcal{I}_{\sigma(T)}^m \quad \text{and} \quad \mathcal{I}_{\mathcal{C}} \not\subseteq \mathcal{I}_{\sigma(T)}^{m+1},$$

where  $\mathcal{I}_{\sigma(T)}$  is the ideal sheaf of  $\sigma(T)$  in  $\mathcal{O}_{T \times S}$ .

**Remark 2**

Note that the above notion commutes with base change, i.e. if we have an equimultiple embedded family of curves in  $S$  over  $T$  as above and if  $\alpha : T' \rightarrow T$  is a morphism, then the fibre product diagram

$$\begin{array}{ccc} T' \times S & \xrightarrow{\quad} & T \times S \\ & \swarrow \quad \searrow & \\ & \mathcal{C}' & \mathcal{C} \\ & \downarrow \varphi' \quad \downarrow \sigma & \downarrow \sigma \quad \downarrow \varphi \\ T' & \xrightarrow{\quad \alpha \quad} & T \end{array}$$

gives rise to an embedded equimultiple family of curves over  $T'$  of the same multiplicity, since locally it is defined via the tensor product.

**Example 3**

Let us denote by  $T_\varepsilon = \text{Spec}(\mathbb{C}[\varepsilon])$  with  $\varepsilon^2 = 0$ . Then a family of curves in  $S$  over  $T_\varepsilon$  is just a Cartier divisor of  $T_\varepsilon \times S$ , that is, it is given on a suitable open covering  $S = \bigcup_{\lambda \in \Lambda} U_\lambda$  by equations

$$f_\lambda + \varepsilon \cdot g_\lambda \in \mathbb{C}[\varepsilon] \otimes_{\mathbb{C}} \Gamma(U_\lambda, \mathcal{O}_S) = \Gamma(U_\lambda, \mathcal{O}_{T \times S}),$$

which glue together to give a global section  $\left\{ \frac{g_\lambda}{f_\lambda} \right\}_{\lambda \in \Lambda}$  in  $H^0(C, \mathcal{O}_C(C))$ , where  $C$  is the curve defined locally by the  $f_\lambda$  (see e.g. [Mum66], Lecture 22).

A section of the family through  $p$  is locally in  $p$  given as  $(x, y) \mapsto (x_a, y_b) = (x + \varepsilon \cdot a, y + \varepsilon \cdot b)$  for some  $a, b \in \mathbb{C}\{x, y\} = \mathcal{O}_{S,p}$ .

**Example 4**

Let  $H$  be a connected component of the Hilbert scheme  $\text{Hilb}_S$  of curves in  $S$ , then  $H$  comes with a universal family

$$\pi : \mathcal{H} \longrightarrow H : (C, p) \mapsto C. \tag{1}$$

Let us now fix a positive integer  $m$  and set

$$\mathcal{H}_m = \{(C, p) \in H \times S \mid C \in H, \text{mult}_p(C) = m\}.$$

Then  $\mathcal{H}_m$  is a locally closed subvariety of  $H \times S$ , and (1) induces via base change a flat and proper family  $\mathcal{F}_m = \{(C_p, q) \in \mathcal{H}_m \times S \mid C_p = (C, p) \in \mathcal{H}_m, q \in C\}$  which has a distinguished section  $\sigma$

$$\begin{array}{ccc} \mathcal{F}_m & \hookrightarrow & \mathcal{H}_m \times S \\ \downarrow \sigma & \swarrow & \\ \mathcal{H}_m & & \end{array} \quad (2)$$

sending  $C_p = (C, p)$  to  $(C_p, p) \in \mathcal{F}_m$ . Moreover, this family is equimultiple along  $\sigma$  of multiplicity  $m$  by construction.

### Example 5

Similarly, if  $|L|$  is a linear system on  $S$ , then it induces a universal family

$$\pi : \mathcal{L} = \{(C, p) \in |L| \times S \mid p \in C\} \longrightarrow |L| : (C, p) \mapsto C. \quad (3)$$

If we now fix a positive integer  $m$  and set

$$\mathcal{L}_m = \{(C, p) \in |L| \times S \mid C \in |L|, \text{mult}_p(C) = m\}.$$

Then  $\mathcal{L}_m$  is a locally closed subvariety of  $|L| \times S$ , and (3) induces via base change a flat and proper family  $\mathcal{G}_m = \{(C_p, q) \in \mathcal{L}_m \times S \mid C_p = (C, p) \in \mathcal{L}_m, q \in C\}$  which has a distinguished section  $\sigma$

$$\begin{array}{ccc} \mathcal{G}_m & \hookrightarrow & \mathcal{L}_m \times S \\ \downarrow \sigma & \swarrow & \\ \mathcal{L}_m & & \end{array} \quad (4)$$

sending  $C_p = (C, p)$  to  $(C_p, p) \in \mathcal{G}_m$ . Moreover, this family is equimultiple along  $\sigma$  of multiplicity  $m$  by construction.

We may interpret  $\mathcal{L}_m$  as the family of curves in  $|L|$  with  $m$ -fold points together with a section which distinguishes the  $m$ -fold point. This is important if the  $m$ -fold point is not isolated or if it splits in a neighbourhood into several simpler  $m$ -fold points.

Of course, since (3) can be viewed as a subfamily of (1) we may view (4) in the same way as a subfamily of (2).

**Definition 6**

Let  $t_0 \in T$  be a pointed complex space,  $C \subset S$  a curve, and  $p \in C$  a point of multiplicity  $m$ . Then an *embedded (equimultiple) deformation of  $C$  in  $S$  over  $t_0 \in T$  with section  $\sigma$  through  $p$*  is a commutative diagram of morphisms

$$\begin{array}{ccccc}
 & & S & \xrightarrow{\quad} & T \times S \\
 & \nearrow & \downarrow & \nearrow & \downarrow \\
 p \in C & \xrightarrow{\quad} & C & \xrightarrow{\quad} & C \\
 \downarrow \sigma & & \downarrow \sigma & \nearrow \varphi & \downarrow \\
 t_0 & \xrightarrow{\quad} & T & & 
 \end{array}$$

where the right hand part of the diagram is an embedded (equimultiple) family of curves in  $S$  over  $T$  with section  $\sigma$ . Sometimes we will simply write  $(\varphi, \sigma)$  to denote a deformation as above.

Given two deformations, say  $(\varphi, \sigma)$  and  $(\varphi', \sigma')$ , of  $C$  over  $t_0 \in T$  as above, a morphism of these deformations is a morphism  $\psi : C' \rightarrow C$  which makes the obvious diagram commute:

$$\begin{array}{ccccc}
 T \times S & \xlongequal{\quad} & T \times S & & \\
 \swarrow & & \searrow & & \\
 C' & \xrightarrow{\quad \psi \quad} & C & & \\
 \downarrow \varphi' & \nearrow & \downarrow \varphi & \nearrow & \\
 T & \xlongequal{\quad} & T & & \\
 \downarrow \sigma' & & \downarrow \sigma & & \\
 t_0 & & t_0 & & 
 \end{array}$$

This gives rise to the deformation functor

$$\underline{\text{Def}}_{p \in C/S}^{\text{sec}, \text{em}} : (\text{pointed complex spaces}) \rightarrow (\text{sets})$$

of embedded equimultiple deformations of  $C$  with section through  $p$  from the category of pointed complex spaces into the category of sets,

where for a pointed complex space  $t_0 \in T$

$$\begin{aligned} \underline{\text{Def}}_{p \in C/S}^{\text{sec}, \text{em}}(t_0 \in T) = \{ & \text{isomorphism classes of embedded equimultiple} \\ & \text{deformations } (\varphi, \sigma) \text{ of } C \text{ in } S \text{ over } t_0 \in T \\ & \text{with section through } p \}. \end{aligned}$$

Moreover, forgetting the section we have a natural transformation

$$\underline{\text{Def}}_{p \in C/\Sigma}^{\text{sec}, \text{em}} \longrightarrow \underline{\text{Def}}_{C/\Sigma}, \quad (5)$$

where the latter is the deformation functor

$$\underline{\text{Def}}_{C/\Sigma} : (\text{pointed complex spaces}) \rightarrow (\text{sets})$$

of embedded deformations of  $C$  in  $S$  given by

$$\underline{\text{Def}}_{C/S}(t_0 \in T) = \{ \text{isomorphism classes of embedded deformations} \\ \text{of } C \text{ in } S \text{ over } t_0 \in T \}.$$

### Example 7

According to Example 3 a deformation of  $C$  in  $S$  over  $T_\varepsilon$  along a section through  $p$  is given by

- local equations  $f + \varepsilon \cdot g$  such that  $f$  is a local equation for  $C$  and the  $\frac{g}{f}$  glue to a global section of  $\mathcal{O}_C(C)$ ,
- together with a section which in local coordinates in  $p$  is given as  $\sigma : (x, y) \mapsto (x_a, y_b) = (x + \varepsilon \cdot a, y + \varepsilon \cdot b)$  for some  $a, b \in \mathbb{C}\{x, y\}$ .

If we forget the section it is well known (see e.g. [Mum66], Lecture 22) that two such deformations are isomorphic if and only if they induce the same global section of  $\mathcal{O}_C(C)$  and this one-to-one correspondence is functorial so that we have an isomorphism of vector spaces

$$\underline{\text{Def}}_{C/S}(T_\varepsilon) \xrightarrow{\cong} H^0(C, \mathcal{O}_C(C)).$$

Considering the natural transformation from (5) we may now ask what the image of  $\underline{\text{Def}}_{p \in C/S}^{\text{sec}, \text{em}}(T_\varepsilon)$  in  $H^0(C, \mathcal{O}_C(C))$  is. These are, of course, the sections which allow a section  $\sigma$  through  $p$  along which the deformation is equimultiple, and according to Lemma 8 we thus have an epimorphism

$$\underline{\text{Def}}_{p \in C/S}^{\text{sec}, \text{em}}(T_\varepsilon) \twoheadrightarrow H^0(C, \mathcal{J}_{Z/C}(C)),$$

where  $\mathcal{J}_{Z/C}$  is the restriction to  $C$  of the ideal sheaf  $\mathcal{J}_Z$  on  $S$  given by

$$\mathcal{J}_{Z,q} = \begin{cases} \mathcal{O}_{S,q}, & \text{if } q \neq p, \\ \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle + \langle x, y \rangle^m, & \text{if } q = p, \end{cases} \quad (6)$$

here  $f$  is a local equation for  $C$  in local coordinates  $x$  and  $y$  in  $p$ .

It remains the question what the dimension of the kernel of this map is, that is, how many different sections such an isomorphism class of embedded deformations of  $C$  in  $S$  over  $T_\varepsilon$  through  $p$  can admit.

J. Wahl showed in [Wah74], Proposition 1.9, that locally the equimultiple deformation admits a unique section if and only if  $C$  in  $p$  is not unitangential. If  $C$  is unitangential we may assume that locally in  $p$  it is given by  $f = y^m + h.o.t..$  If we have an embedded deformation of  $C$  in  $S$  which along some section is equimultiple of multiplicity  $m$ , then locally it looks like

$$f + \varepsilon \cdot \left( a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} + h \right)$$

with  $h \in \langle x, y \rangle^m$ . However, since  $\frac{\partial f}{\partial x} \in \langle x, y \rangle^m$  the deformation is equimultiple along the sections  $(x, y) \mapsto (x + \varepsilon \cdot (c + a), y + \varepsilon \cdot b)$  for all  $c \in \mathbb{C}$ . Thus in this case the kernel turns out to be one-dimensional, i.e. there is a one-dimensional vector space  $\mathcal{K}$  such that the following sequence is exact:

$$0 \rightarrow \mathcal{K} \rightarrow \underline{\text{Def}}_{p \in C/S}^{\text{sec}, \text{em}}(T_\varepsilon) \rightarrow H^0(C, \mathcal{J}_{Z/C}(C)) \rightarrow 0. \quad (7)$$

### Lemma 8

Let  $f + \varepsilon \cdot g$  be a first-order infinitesimal deformation of  $f \in \mathbb{C}\{x, y\}$ ,  $m = \text{ord}(f)$ ,  $a, b \in \mathbb{C}\{x, y\}$ , and  $x_a = x + \varepsilon \cdot a$ ,  $y_b = y + \varepsilon \cdot b$ .

Then  $f + \varepsilon \cdot g$  is equimultiple along the section  $(x, y) \mapsto (x_a, y_b)$  if and only if

$$g - a \cdot \frac{\partial f}{\partial x} - b \cdot \frac{\partial f}{\partial y} \in \langle x, y \rangle^m.$$

In particular,  $f + \varepsilon \cdot g$  is equimultiple along some section if and only if

$$g \in \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle + \langle x, y \rangle^m.$$

**Proof:** If  $a, b \in \mathbb{C}\{x, y\}$  and  $h \in \langle x, y \rangle^m$  then by Taylor expansion and since  $\varepsilon^2 = 0$  we have

$$f + \varepsilon \cdot \left( a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} + h \right) = f(x_a, y_b) + \varepsilon \cdot h(x_a, y_b),$$

where  $f(x_a, y_b), h(x_a, y_b) \in \langle x_a, y_b \rangle^m$ , i.e. the infinitesimal deformation  $f + \varepsilon \cdot \left( a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} + h \right)$  is equimultiple along  $(x, y) \mapsto (x_a, y_b)$ .

Conversely, if  $f + \varepsilon \cdot g$  is equimultiple along  $(x, y) \mapsto (x_a, y_b)$  then

$$f(x, y) + \varepsilon \cdot g(x, y) = F(x_a, y_b) + \varepsilon \cdot G(x_a, y_b)$$

with  $F(x_a, y_b), G(x_a, y_b) \in \langle x_a, y_b \rangle^m$ . Again, by Taylor expansion and since  $\varepsilon^2 = 0$  we have

$$f(x, y) = f(x_a, y_b) - \varepsilon \cdot \left( a \cdot \frac{\partial f}{\partial x}(x_a, y_b) + b \cdot \frac{\partial f}{\partial y}(x_a, y_b) \right)$$

and

$$\varepsilon \cdot g(x, y) = \varepsilon \cdot g(x_a, y_b).$$

Thus

$$F(x_a, y_b) = f(x_a, y_b)$$

and

$$\langle x_a, y_b \rangle^m \ni G(x_a, y_b) = g(x_a, y_b) - a \cdot \frac{\partial f}{\partial x}(x_a, y_b) - b \cdot \frac{\partial f}{\partial y}(x_a, y_b).$$

□

### Example 9

If we fix a curve  $C \subset S$  and a point  $p \in C$  such that  $\text{mult}_p(C) = m$ , i.e. if using the notation of Example 4 we fix a point  $C_p = (C, p) \in \mathcal{H}_m$ , then the diagram

$$\begin{array}{ccccc}
 S & \xrightarrow{\cong} & \{C_p\} \times S & \hookrightarrow & \mathcal{H}_m \times S \\
 \uparrow & & \uparrow & & \nearrow \\
 C & \xrightarrow{\cong} & \{(C_p, q) \mid q \in C\} & \hookrightarrow & \mathcal{F}_m \\
 \downarrow & & \downarrow & \searrow \sigma & \downarrow \\
 t_0 & \xlongequal{\quad} & C_p & \hookrightarrow & \mathcal{H}_m
 \end{array} \tag{8}$$

is an embedded equimultiple deformation of  $C$  in  $S$  along the section  $\sigma$  through  $p$ . Moreover, any embedded equimultiple deformation of  $C$  in  $S$  with section through  $p$  as a family is up to isomorphism induced via

(1) in a unique way and thus factors obviously uniquely through (8). This means that every equimultiple deformation of  $C$  in  $S$  through  $p$  is induced up to isomorphism in a unique way from (8).

We now want to examine the tangent space to  $\mathcal{H}_m$  at a point  $C_p = (C, p)$ , which is just

$$T_{C_p}(\mathcal{H}_m) = \mathrm{Hom}_{\mathrm{loc-K-Alg}}(\mathcal{O}_{\mathcal{H}_m, C_p}, \mathbb{C}[\varepsilon]) = \mathrm{Hom}(T_\varepsilon, (\mathcal{H}_m, C_p)),$$

where  $(\mathcal{H}_m, C_p)$  denotes the germ of  $\mathcal{H}_m$  at  $C_p$ . However, a morphism

$$\psi : T_\varepsilon \longrightarrow (\mathcal{H}_m, C_p)$$

gives rise to a commutative fibre product diagram

$$\begin{array}{ccccc} T_\varepsilon \times_{\mathcal{H}_m} \mathcal{F}_m & \longrightarrow & \mathcal{F}_m & & \\ \sigma' \uparrow & & \downarrow \varphi' & & \downarrow \sigma \\ T_\varepsilon & \xrightarrow{\psi} & \mathcal{H}_m & & \end{array}$$

sending the closed point of  $T_\varepsilon$  to  $C$ . Thus  $(\varphi', \sigma') \in \underline{\mathrm{Def}}_{p \in C/S}^{\mathrm{sec}, \mathrm{em}}(T_\varepsilon)$  is an *embedded equimultiple deformation of  $C$  in  $S$  with section through  $p$* . The universality of (8) then implies that up to isomorphism each one is of this form for a unique  $\varphi'$ , and this construction is functorial. We thus have

$$T_{C_p}(\mathcal{H}_m) \cong \underline{\mathrm{Def}}_{p \in C/S}^{\mathrm{sec}, \mathrm{em}}(T_\varepsilon),$$

and hence (7) gives the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow T_{C_p}(\mathcal{H}_m) \longrightarrow H^0(C, \mathcal{J}_{Z/C}(C)) \longrightarrow 0.$$

In particular,

$$\dim_{\mathbb{C}}(T_{C_p}(\mathcal{H}_m)) = \begin{cases} \dim_{\mathbb{C}} H^0(C, \mathcal{J}_{Z/C}(C)) - 2, & \text{if } C \text{ is unitangential,} \\ \dim_{\mathbb{C}} H^0(C, \mathcal{J}_{Z/C}(C)) - 1, & \text{else.} \end{cases}$$

### Example 10

If we do the same constructions replacing in (8) the family (2) by (4) we get for the tangent space to  $\mathcal{L}_m$  at  $C_p = (C, p)$  the diagram of exact



sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & T_{C_p}(\mathcal{H}_m) & \longrightarrow & H^0(C, \mathcal{J}_{Z/C}(C)) \longrightarrow 0 \\
& & \parallel & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & T_{C_p}(\mathcal{L}_m) & \longrightarrow & H^0(S, \mathcal{J}_Z(C))/H^0(\mathcal{O}_S) \longrightarrow 0.
\end{array}$$

In order to see this consider the exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{O}_C(C) \rightarrow 0$$

induced from the structure sequence of  $C$ . This sequence shows that the tangent space to  $|L|$  at  $C$  considered as a subspace of the tangent space  $H^0(C, \mathcal{O}_C(C))$  of  $H$  at  $C$  is just  $H^0(S, \mathcal{O}_S(C))/H^0(S, \mathcal{O}_S)$  – that is, a global section of  $\mathcal{O}_C(C)$  gives rise to an embedded deformation of  $C$  in  $S$  which is actually a deformation in the linear system  $|L|$  if and only if it comes from a global section of  $\mathcal{O}_S(C)$ , and the constant sections induce the trivial deformations. This construction carries over to the families (2) and (4).

In particular we get the following proposition.

**Proposition 11**

*Using the notation from above let  $C$  be a curve in the linear system  $|L|$  on  $S$  and suppose that  $p \in C$  such that  $\text{mult}_p(C) = m$ .*

*Then the tangent space of  $\mathcal{L}_m$  at  $C_p = (C, p)$  satisfies*

$$\dim_{\mathbb{C}}(T_{C_p}(\mathcal{L}_m)) = \begin{cases} \dim_{\mathbb{C}} H^0(S, \mathcal{J}_Z(C)) - 2, & \text{if } C \text{ is unitangential,} \\ \dim_{\mathbb{C}} H^0(S, \mathcal{J}_Z(C)) - 1, & \text{else.} \end{cases}$$

*Moreover, the expected dimension of  $T_{C_p}(\mathcal{L}_m)$  and thus of  $\mathcal{L}_m$  at  $C_p$  is just*

$$\text{expdim}_{C_p}(\mathcal{L}_m) = \text{expdim}_{\mathbb{C}}(T_{C_p}(\mathcal{L}_m)) = \dim |L| - \frac{(m+1) \cdot m}{2} + 2.$$

For the last statement on the expected dimension just consider the exact sequence

$$0 \rightarrow H^0(S, \mathcal{J}_Z(C)) \rightarrow H^0(S, \mathcal{O}_S(L)) \rightarrow H^0(S, \mathcal{O}_Z)$$

and note that the dimension of  $H^0(S, \mathcal{J}_Z(C))$ , and hence of  $T_{C_p}(C)$ , attains the minimal possible value if the last map is surjective. The

expected dimension of  $H^0(S, \mathcal{J}_Z(C))$  hence is

$$\expdim_{\mathbb{C}} H^0(S, \mathcal{J}_Z(C)) = \dim |L| + 1 - \deg(Z),$$

and it suffices to calculate  $\deg(Z)$ . If  $C$  is unitangential we may assume that  $C$  locally in  $p$  is given by  $f = y^m + h.o.t.$ , so that

$$\mathcal{O}_{Z,p} = \mathbb{C}\{x, y\} / \langle y^{m-1} \rangle + \langle x, y \rangle^m,$$

and hence  $\deg(Z) = \frac{(m+1) \cdot m}{2} - 1$ . If  $C$  is not unitangential, then we may assume that it locally in  $p$  is given by an equation  $f$  such that  $f_m = \text{jet}_m(f) = x^\mu \cdot y^\nu \cdot g$ , where  $x$  and  $y$  do not divide  $g$ , but  $\mu$  and  $\nu$  are at least one. Suppose now that the partial derivatives of  $f_m$  are not linearly independent, then we may assume  $\frac{\partial f_m}{\partial x} \equiv \alpha \cdot \frac{\partial f_m}{\partial y}$  and thus

$$\mu y g \equiv \alpha \nu x g + \alpha x y \cdot \frac{\partial g}{\partial y} - x y \cdot \frac{\partial g}{\partial x},$$

which would imply that  $y$  divides  $g$  in contradiction to our assumption. Thus the partial derivatives of  $f_m$  are linearly independent, which shows that

$$\deg(Z) = \dim_{\mathbb{C}} \left( \mathbb{C}\{x, y\} / \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle + \langle x, y \rangle^m \right) = \frac{(m+1) \cdot m}{2} - 2.$$

### Example 12

Let us consider the Example 5 in the case where  $S = \mathbb{P}^2$  and  $L = \mathcal{O}_{\mathbb{P}^2}(d)$ . We will show that  $\mathcal{L}_m$  is then *smooth of the expected dimension*. Note that  $\pi(\mathcal{L}_m)$  will only be smooth at  $C$  if  $C$  has an ordinary  $m$ -fold point, that is, if all tangents are different.

Given  $C_p = (C, p) \in \mathcal{L}_m$  we may pass to a suitable affine chart containing  $p$  as origin and assume that  $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$  is parametrised by polynomials

$$F_{\underline{a}} = f + \sum_{i+j=0}^d a_{i,j} \cdot x^i y^j,$$

where  $f$  is the equation of  $C$  in this chart. The closure of  $\pi(\mathcal{L}_m)$  in  $|L|$  locally at  $C$  is then given by several equations, say  $F_1, \dots, F_k \in \mathbb{C}[a_{i,j} | i+j=0, \dots, d]$ , in the coefficients  $a_{i,j}$ . We get these equations by eliminating the variables  $x$  and  $y$  from the ideal defined by

$$\left\langle \frac{\partial^{i+j} F_{\underline{a}}}{\partial x^i \partial y^j} \mid i+j=0, \dots, m-1 \right\rangle.$$

And  $\mathcal{L}_m$  is locally in  $C_p$  described by the equations

$$F_1 = 0, \dots, F_k = 0, \quad \frac{\partial^{i+j} F_a}{\partial x^i y^j} = 0, \quad i + j = 0, \dots, m - 1.$$

However, the Jacoby matrix of these equations with respect to the variables  $x, y, a_{i,j}$  contains a diagonal submatrix of size  $\frac{m \cdot (m+1)}{2}$  with ones on the diagonal, so that its rank is at least  $\frac{m \cdot (m+1)}{2}$ , which – taking into account that  $|L| = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)))$  – implies that the tangent space to  $\mathcal{L}_m$  at  $C_p$  has codimension at least  $\frac{m \cdot (m+1)}{2} - 1$  in the tangent space of  $\mathcal{L}$ . By Proposition 11 we thus have

$$\begin{aligned} \dim_{C_p}(\mathcal{L}_m) &\leq \dim_{\mathbb{C}} T_{C_p}(\mathcal{L}_m) \leq \dim_{\mathbb{C}} T_{C_p}(\mathcal{L}) - \frac{m \cdot (m+1)}{2} + 1 \\ &= \dim(\mathcal{L}) - \frac{m \cdot (m+1)}{2} + 1 \\ &= \dim |L| - \frac{m \cdot (m+1)}{2} + 2 \\ &= \expdim_{C_p}(\mathcal{L}_m) \leq \dim_{C_p}(\mathcal{L}_m), \end{aligned}$$

which shows that  $\mathcal{L}_m$  is smooth at  $C_p$  of the expected dimension.

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